# Hex is PSPACE-complete

Stefan Reisch
Universität Bielefeld, Fakultät für Mathematik, Bielefeld, Germany

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**Summary.** There are a number of board games such as Checkers [2], Go [5], and Gobang [8], which are known to be PSPACE-hard. This means that the problem to determine the player having a winning strategy in a given situation on an $n \times n$ board of one of these games is as hard to solve as any problem computable in polynomial space. PSPACE-completeness has been previously proven for some combinatorial games played on graphs or by logical formulas [1, 9].

In this paper we will show that the same holds for the game of Hex. The crucial point of the proof is to establish PSPACE-hardness for a generalization of Hex played on planar graphs. This will be done by showing that the problem of deciding whether a given quantified boolean formula in conjunctive normal form is true, is polynomial time-reducible to the decision problem for generalized Hex. In order to do this we will use methods which were also used to prove PSPACE-completeness of planar Geography in [5]. Therefore our proof is quite different from the proof provided by Even and Tarjan [1], who showed PSPACE-completeness of generalized Hex played on arbitrary graphs. Since it is easy to see that the decision problem for Hex is in PSPACE, the decision problem for Hex is PSPACE-complete.

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The proof presented here continues in the style of proofs for other board games. It will be shown that for the game of Hex, deciding which player has a winning strategy in a given situation is PSPACE-hard. Similar proofs are available for the games of Go[1], Checkers[2], and Gobang[3]. For Hex it is furthermore true that determining which player has a winning strategy is in

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[1] see [5]
[2] see [2]
[3] see [8]

PSPACE, and is thus PSPACE-complete.

Hex is played on a rhombus-shaped grid, as shown in Fig. 1. The game, invented in 1942 by the Danish physicist Piet Hein, is played according to the following rules: two players, called "white" and "black" here, take turns placing stones of their color on the grid points of the playing field. Opposing pairs of edges of the playing field are assigned to each player, as indicated by the labels in Fig. 1. The goal of each player is to connect his two edges with a chain of stones of his color, and to block his opponent from connecting his two sides. Placed stones may not be removed nor moved, and no grid point may be occupied twice. The size of the playing field varies, normally comprising $11 \times 11$ grid points, as shown in Fig. 1. When an edge of the playing field is $n$ grid points long, we will refer to the game as $n \times n$ Hex.
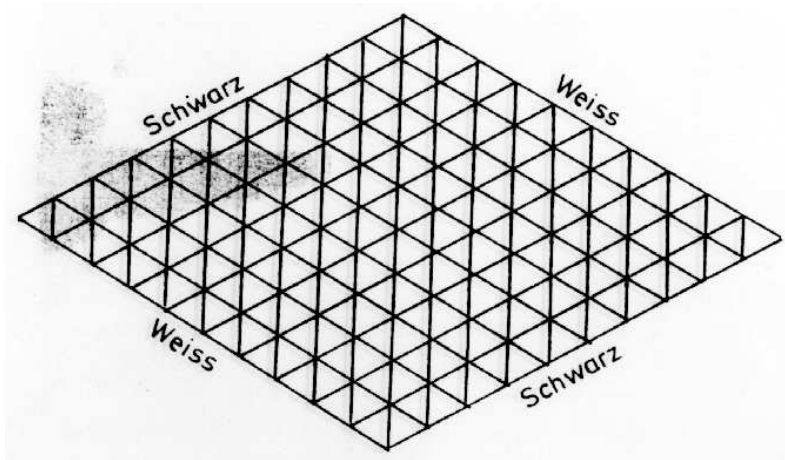


Figure 1: Schwarz=black; Weiss=white

The following theorem can be easily proved:

**Fact 0.1** *For all n: on an empty playing field, the first player has a winning strategy in $n \times n$ Hex.* [4]

We will be considering playing situations in $n \times n$ Hex, where various points on the playing field are occupied with white and black stones. Our main result will be that determining which player has a winning strategy in a given playing situation is PSPACE-complete.

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[4]for proof, see [3]

# I

Let $A$ be a finite alphabet. We say that a function $f : A^* \to A^* \left( A^* = \bigcup_{i=0}^{\infty} A^n \right)$
is computable in polynomial space (or time) if there exists a determinis-
tic $O(n^k)$ tape-length-limited (or, respectively, time-limited) Turing machine
which calculates $f$. That is, for an input $x \in A^*$, $f(x)$ can be calculated
using a $O(|x|^k)$-length-limited portion of tape (or, respectively, in $O(|x|^k)$
processing steps).

PSPACE (or P) refers to the class of languages $L \subset A^*$, whose characteris-
tic function $ch_L : A^* \to \{0, 1\}$ ($ch_L(x) = 1 \Leftrightarrow x \in L$) is computable in polynomial
space (or time, respectively). A language $L_1$ is said to be polynomial time
reducible to a language $L_2$, notated as $L_1 \leq_p L_2$, when there exists a function
$f : A^* \to A^*$ computable in polynomial time, and $f(x) \in L_2 \Leftrightarrow x \in L$. That is,
the problem of determining whether $x \in L_1$ is "relatively easily" converted
into the analogous problem for $L_2$. [5]

If $L_0 \subset A^*$, and for all $L \in$ PSPACE: $L \leq_p L_0$, then $L_0$ is termed PSPACE-
hard. A language $L \subset A^*$ is PSPACE-complete if $L \in$ PSPACE and $L$ is
PSPACE-hard.


# II

We want to consider the decision problem for the following language.

$n \times n$ Hex situations should be encoded in a suitable manner by an Al-
phabet $A$; that is, the language of words that encode play situations should
be in P. In particular, the following subset of $A^*$ forms a language:

> $G(\text{Hex}) := \{a \in A^* \mid a$ is an encoding of a play situation in an
> $n \times n$ Hex game, in which the player on the move ("white") has a
> winning strategy.$\}$

We want to show the PSPACE-completeness of this language.

In addtion, we consider the following decision problem and corresponding
language:

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[5]along with polynomial-time reducibility, the stronger logarithmic tape length re-
ducibility is often discussed; but we won't be using it here

CNF refers to the set of boolean formulas

$$f(x_1, \ldots, x_n) = \bigwedge_{k=1}^{m} \left( \bigvee_{j=1}^{l_k} x_{k,j}{}^{c_{k,j}} \right)$$

in conjunctive normal form.

$$(c_{k,j} \in \{0,1\}; \; x^0 = \neg x, \; x^1 = x).$$

The set of boolean expressions $F = Q_1 x_1 \ldots Q_n x_n f(x_1, \ldots, x_n)$ ($Q_i \in \{\exists, \forall\}$) can be understood as a language on an alphabet $A$. Let

$$Q := \{F = Q_1 x_1 \ldots Q_n x_n f(x_1, \ldots, x_n) \mid f \in \mathit{CNF}, \; Q_i \in \{\exists, \forall\} (1 \le i \le n)\}$$

and:

$$Q_e := \{F \in Q \mid F \text{ is true}\}$$

The following theorem is true:

**Fact II.1** $Q_e$ *is PSPACE-complete.*[6]

For the language $G(\mathrm{Hex})$ we want to prove the theorem:

**Theorem II.2** $Q_e \le_p G(\text{Hex})$.

From this theorem follows:

**Corollary II.3** $G(\text{Hex})$ *is PSPACE-complete.*

From the transitivity of the relation "$\le_p$", it follows from theorems II.1 and II.2 that $G(\mathrm{Hex})$ is PSPACE-hard. Furthermore, it's easy to see that $G(\mathrm{Hex}) \in \mathrm{PSPACE}$, as the duration of play is limited by the count of open grid points; that is, it is polynomially limited. At each move, each player on an $n \times n$ board chooses from at most $n^2$ different possible moves. The proof of Corollary II.3 follows the line of reasoning by which it is generally proved, that the decision problem for various combinatorial games is in PSPACE. Details of such proofs can be found in [1,9] or in [7], p. 209ff.

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[6]for proof, see [6]

The crux of the proof of Theorem II.2 will be in the proof of a lemma which deals with a generalization of Hex to undirected graphs. Hex can be generalized to a game on undirected graphs in the following manner:

Given is an undirected, planar, and finite graph $G = (V, E)$ with two selected points, named $s$ and $t$. These two points form an *outside pair* of the graph, that is the graph $G = (V, E \cup \{s, t\})$ remains planar. Such an outside pair has the characteristic that the corresponding graph can be embedded in the plane (the space $\mathbb{R}^2$) so that both points $s$ and $t$ are "accessible from the outside". We will hereafter refer to an planar graph with a selected outside point pair as a Hex-graph.

On a Hex graph, Hex can be played according to the following rules: two players take turns placing stones on the nodes of the graph (except for $s$ and $t$). One player places white stones; the other black. White plays first on an empty board; his goal is to place white stones on a complete path between $s$ and $t$. The points $s$ and $t$ themselves are not played on. If "white" succeeds in making the path, he has won; if stones placed by "black" prevent such a path, black has won, as no stone may be moved or removed, and no point may be played on twice. We will call this Hex game on graphs "graph-Hex".

We will consider game situations in graph-Hex, that is, we require that individual points on the Hex-graph have black or white stones placed on them.

As is the case with game situations in $n \times n$ Hex, play situations in graph-Hex can be encoded by an alphabet $A$, so that the language of words which encode graph-Hex play situations is in P.

We define the following language:

$G(\text{graph-Hex}) := \{a \in A^* \mid a$ is an encoding of a play situation in graph-Hex, which satisfies conditions (II.1) and (II.2).$\}$

(II.1) The player on the move ("white") has a winning strategy.
(II.2) Each open point in the play situation is incident to at most three edges in the Hex-Graph.

**(Main-)Lemma II.4**

$$Q_e \leq_p G(graph\text{-}Hex).$$

In [1], Even and Tarjan give a similar theorem for general, nonplanar graphs, namely that deciding which player has a winning strategy at the start of a graph-Hex game with two selected points $s$ and $t$ is PSPACE-complete.

# III

The proof for lemma II.4 will be done in stages. Its basic method follows the proof of the theorem that generalized "Geography", played on a bipartite planar graph, is PSPACE-complete.

Generalized "Geography" is played on the nodes of a directed graph, which has a selected point $s$. Two players alternately place stones on the nodes of a graph, where the first player starts by playing on the selected point $s$, and on subsequent turns play may only occur on nodes pointed to by an arc that starts on the most recently played-upon node. The graph that the game is played on should meet the following requirements:

(III.1) The graph $G = (V, E)$ should be planar and bipartite. (The latter means that the set of nodes $V$ can be decomposed: $V = V_1 \cup V_2$; $V_1 \cap V_2 = \emptyset$ and $E \subset (V_1 \times V_2) \cup (V_2 \times V_1)$ ).
(III.2) Each node $v \in V$ of the graph $G = (V, E)$ is incident to at most three edges, that is $id(s) + od(s) \leq 3$. And in particular, for all nodes $v \in V$: $od(v) \neq 0$; and for the point s: $id(s) = 0$, and $od(s) = 1$.

The player who has no more possible moves to make has lost the game. From the rules of the game and the fact that the graph being played on is bipartite, it immediately follows that the set of nodes $V$ is decomposable into two subsets $V_1$ and $V_2$ with the consequence that the first player only can play on nodes in $V_1$ and the second player only can play on nodes in $V_2$. A directed graph with the properties (III.1) and (III.2) will be called a "Geography"-graph. Because the requirements (III.1) and (III.2) are usually not imposed, for the sake of clarity we will speak of "Bipartite Geography".
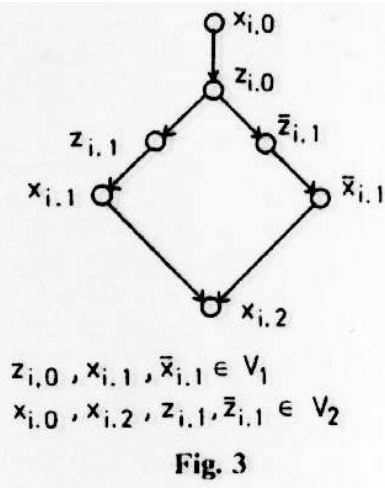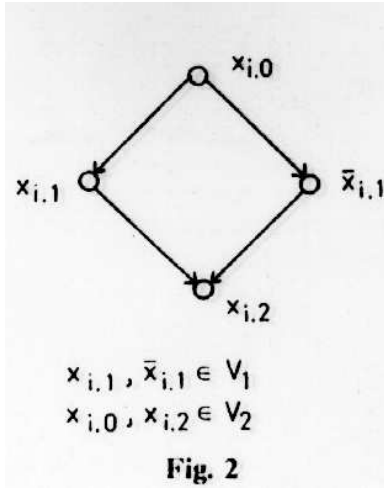
The "Geography"-graphs should be encoded in an alphabet $A$, so that—as was the case in Hex and graph-Hex play situations—the language $L = \{a \in A \mid a$ is an encoding of a "Geography"-graph$\}$ is in P.

Analogous to the definitions of $G(\text{Hex})$ and $G(\text{graph-Hex})$, let

$G(\text{"Bipartite Geography"}) := \{a \in A^* \mid a$ is an encoding of a "Geography"-graph, in which the player to move first in generalized "Geography" has a winning strategy.$\}$

The following fact can be proven using fact II.3:

**Fact III.1** *G("Bipartite Geography") is PSPACE-complete.*

$$x_{i,1} , \bar{x}_{i,1} \in V_1$$
$$x_{i,0} , x_{i,2} \in V_2$$

**Fig. 2**

$$z_{i,0} , x_{i,1} , \bar{x}_{i,1} \in V_1$$
$$x_{i,0} , x_{i,2} , z_{i,1} , \bar{z}_{i,1} \in V_2$$

**Fig. 3**

The proof of the statement that $G$ ("Bipartite Geography") is PSPACE-hard will be done in two steps. First: from a boolean expression $F \in Q$, a directed, not yet planar but bipartite graph is produced $G = (V_1 \cup V_2, E)$. In a second step, the graph is made planar.

Given is a boolean expression $Q_1 x_1 \ldots Q_n x_n f(x_1, \ldots, x_n) = F \in Q$. Each variable $x_i$ in the expression F is represented in "Geography"-graphs by a diamond of the form as shown in figures 2 and 3. Variables associated with an existential quantifier are represented by diamonds as shown in figure 2, and variables associated with a universal quantifier are represented by diamonds as shown in figure 3. These diamonds are linked in a chain, in this case by identifying the points $x_{i,2}$ and $x_{i+1,0}$ with each other.

Each disjunction term in the boolean expression $f(x_1, \ldots, x_n)$

$$f_k^{\vee}(x_{k,1}, \ldots, x_{k,l_k}) = \left( \bigvee_{j=1}^{l_k} x_{k,j}{}^{c_{k,j}} \right)$$

is represented with a node $y_k \in V_2$.

Edges lead from the nodes $y_k$ to the variable nodes $x_{i,1}$ and $\bar{x}_{i,1}$, when the value $x_i$ or $\neg x_i$ (respectively) appears in the disjunctiom term $f_k^{\vee}(\ldots)$. In addition, a node $t \in V_1$ is introduced with edges $(x_{n,2}, t)$ and $(t, y_k)$ to each term-node $y_k$. The game should begin on a node $s \in V_a$, and there is an edge $(s, x_{1,0})$.

A graph constructed in this way consists of two pieces: the chain of diamonds that represent the variables, and a formula portion. An example

of a not-yet-planar "Geography"-graph, constructed according to a boolean expression $F \in Q$, is shown in figure 4, for $F = \exists x_1 \forall x_2 \forall x_3 (x_2 \vee \neg x_3) \wedge (x_1 \vee x_3)$.
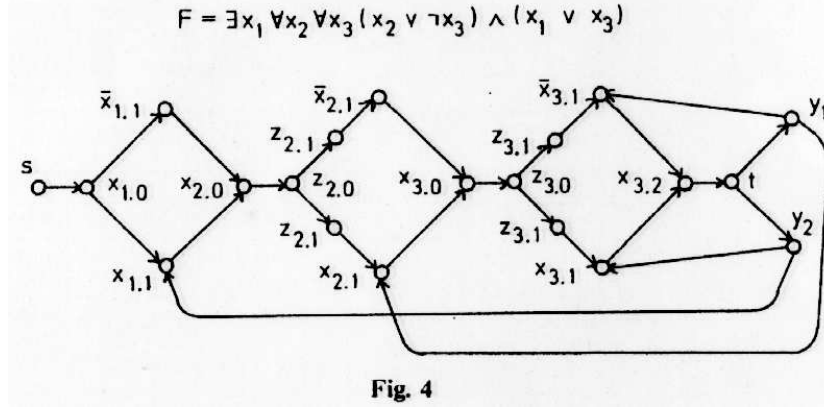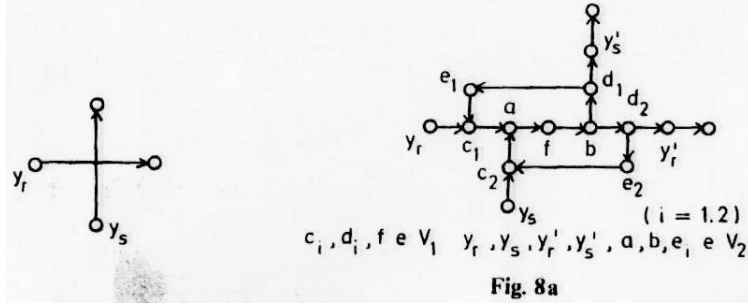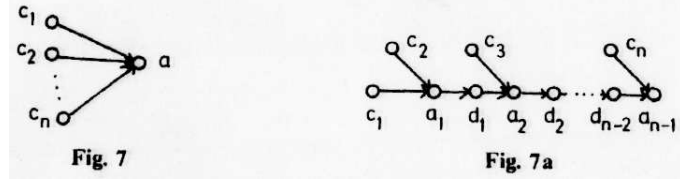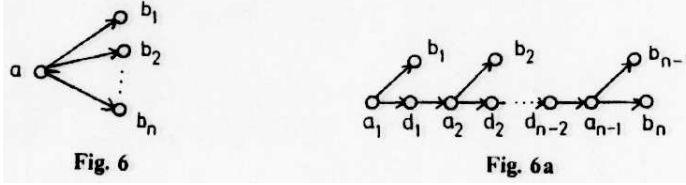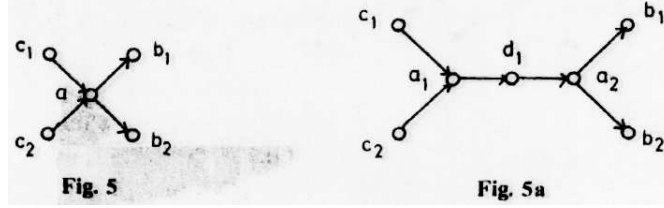


**Fig. 4**

A "Geography"-game on a graph constructed as described is played first in the chain of variable diamonds, and then in the formula portion, from which the play leads back to a single variable diamond. By playing in the variable diamonds, the variables take on specific 0 or 1 values; the variable $x_1$ takes on the value 1 (or 0) when the node $\bar{x}_{i,1}$ (or $x_{i,1}$, respectively) is played on. In subsequent play in the formula portion, the first player (who is also called the $\exists$-player, as he determines the values of the variables bound by existential quantifiers; the other player is then called the $\forall$-player) has a winning strategy when the underlying expression (with specific values set by already-played moves) is true, because only then must every term contain a constant of value 1, thus the second player cannot play on a term-node $y_k$ which has no free predecessor $x_{i,1}$ (or $\bar{x}_{i,1}$, respectively). [7]

Now we come to the second step of the proof. With the help of the following observations, one sees how a true "Geography"-graph can be obtained from the graphs constructed so far.

1) Nodes that are incident to more than three edges can be replaced with subgraphs. The cases shown in Fig. 5, Fig. 6, and Fig. 7 are relevant here. We substitute the subgraphs in Fig. 5, Fig. 6, and Fig. 7 with others as shown in Fig. 5a, 6a, and 7a. In those cases, if $a \in V_j$ $(j = 1, 2)$ $a_i \in V_j$ and $b_i$, $c_i$, $d_i \in V_{3...j}$.

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[7] cf. [7, p.211ff] and [5 and 9]. Note also that what is considered a run-on sentence in English is acceptable style in German.

Fig. 5    Fig. 5a

Fig. 6    Fig. 6a

Fig. 7    Fig. 7a

$c_i, d_i, f \in V_1$   $y_r, y_s, y_r', y_s', a, b, e_i \in V_2$   $(i = 1.2)$

Fig. 8a

2) Only edges emanating from a term node $y_k$ can cross over one another. In play on such a graph, only one of two intersecting edges can be played on, and it may only be played on once; thus intersections as in Fig. 8 can be replaced by subgraphs as in Fig. 8a.

With appropriate introduction of edges, we can achieve: [8]

$$c_i, f, d_i \in V_1 \text{ and } y_r, y_s, a, b, e_i, y_r', y_s' \in V_2$$

.

If a player tries to block play from proceeding from $y_r$ to $y_x'$ or from $y_s$ to $y_x'$, he loses immediately. □

**Remark III.2** *The "Geography"-graphs generated by the procedure described in the proof of fact III.1 have the characteristic that a game played on such a graph is practiaclly decided by the points $x_{i,1}$ or $\bar{x}_{i,1}$ in the variable diamonds*

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[8]For nodes $y_k$ that substitute for terms $f_k^{\vee}(\ldots)$, it is always true that $y_k \in V_2$.

*(see Fig. 2 and Fig. 3), or by point $c_i$ in Fig. 8a; that is, by whether or not the first player can play on such a point. We will call such points decision points.*

# IV

In this section we will show the essential portion of the proof of theorem II.2, the proof of lemma II.4.

**Lemma IV.1** $Q_e \leq_p G(Graph - Hex)$.

To prove this lemma, we need to provide a polynomial-time-limited reduction procedure. This reduction procedure should be composed of three distinct reduction steps:

   a) In the first reduction step, for a given boolean expression $F \in Q$, a planar and bipartite graph is constructed, on which the first player only has a winning strategy when $F$ is true, in accordance with the technique in the proof of fact III.1.

   b) In the second reduction step, the "Geography"-graph constructed from expression $F$ is converted into a playing situation on a (not yet planar, however) 'Hex-graph'. In this playing situation, "white" should be on the move and he should have a winning strategy only in those cases where the first player on the "Geography"-graph has one. The conversion of the "Geography"-graph proceeds by replacing subgraphs around individual points of the "Geography"-graph with specific Hex-graphs, which then become subgraphs of the new Hex-graph. The Hex-graphs are constructed so that these individual subgraphs are played through in sequence in the style of moves in "Geography". Should a player deviate from these informal rules, he risks losing in a few moves. In particular, that means that in a game on a Hex-graph of this kind, "White" can only play such that he lays down a chain of white stones starting at point $s$, whose direction "black" can influence in individual subgraphs. By playing a chain of stones, a "Geography"-game is simulated on Hex-graphs.

   c) In the third reduction step, the graph generated in the first two steps is made into a planar Hex-graph, and in particular, in such a way as to preserve any existing winning strategy for a player.

Whereas the reduction step a) in section III has already been explained sufficiently, we need to explain steps b) and c).

About reduction step b):

First we want to consider some specific playing situations on Hex-graphs. These playing situations have the characteristic that the course of play for the next moves is already determined by the (partial) situations in a small portion of the graph. When we say that "white" has a winning strategy on a graph, we mean specifically that "white" has this winning strategy when he is on the move.

As mentioned, subgraphs in "Geography"-graphs should be substituted with special Hex-graphs. Some of the Hex graphs to be used are described by the illustrations Fig. 9a, 10a, and 11a. (Use the portions remaining after the subgraphs $G_i(i = 1, 2, 3)$ are removed. See Fig. 9b, 10b, and 11b.)

For the playing situations in Fig. 9b, 10b, and 11b, the following statements should hold true:

(IV.1) *"White" is on the move and has to be able to occupy exactly one path between the points $s$ and $t_1$ in the graph $G_1$.*

(IV.2) *Each attempt by "white" to create a path from $s$ to any point in the subgraphs $G_2$ or $G_3$ without wanting to route the path through point $t_1$ can be thwarted by "black" by appropriate countermoves on points in the graph $G_1$. ("Black" is only forced to respond to moves by "white" on points in $G_1$.)*

(IV.3) *"White" can lose within four moves, when he plays on a point in the subgraphs $G_2$ or $G_3$.*

(IV.4) *"Black" can play such that "white" can only play on points in the subgraphs $G_2$ and $G_3$ without risking a quick loss when he has played on a complete path from $s$ into one of these subgraphs. Similarly, "white" can play such that in every response to white's moves on points outside of $G_2$ or $G_3$, "black" must also play on points outside of $G_2$ or $G_3$ (or else "white" wins within two moves).*

(IV.5) *"Black" can play such that "white" can continue his existing chains between points $s$ and $t_1$ only in the graphs $G_2$ or $G_3$. Black can either interrupt a connection in the other partial graphs ($G_2$ or $G_3$) or a connection to $t$ via the point $d$, or play in such a way that any move by "white" which would be necessary for such a connection can be rendered useless by a countermove.*

(IV.6) *In particular, for a game on the graphs in Fig. 9a, Fig. 10a and Fig. 11a:*

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(IV.6.1) *In the graph in Fig. 9a, "white" can choose whether he wants to achieve a connection in subgraph $G_2$ or $G_3$.*
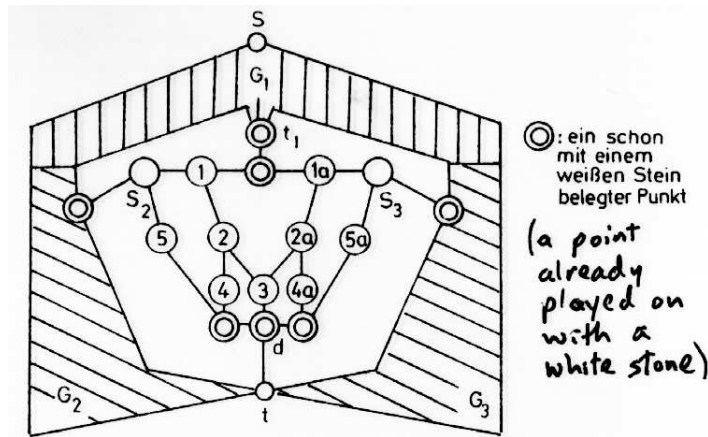


Fig. 9a



Fig. 9b



Fig. 9c
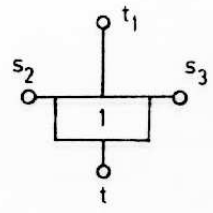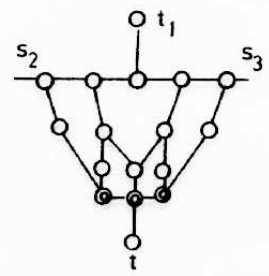
(IV.6.2) *In the graph in Fig. 10a, "black" has the opportunity to determine which of the subgraphs $G_2$ or $G_3$ in which "white" can make a connection.*

(IV.6.3) *In the graph in Fig. 11a, "white" can only make a connection in the subgraph $G_2$; "black" can obstruct a connection to $G_3$.*

While in (IV.1) and (IV.2), assumptions are made about the playing situation in subgraph $G_1$, the statements in (IV.3) through (IV.6) are to be proven.

*About (IV.3):* We assume that "white" plays in a point in the subgraph $G_2$ or $G_3$. "Black" then has the following options:
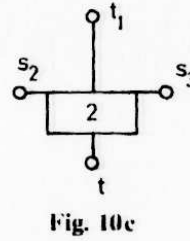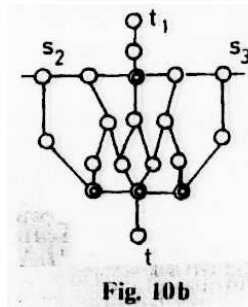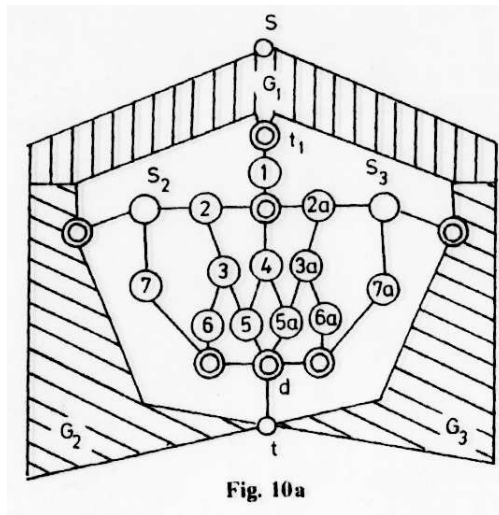
a) In the graph in Fig. 9a, "black" can win after a move to point ③. He only needs to respond to white's moves as follows:

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| Move by "white": | | ① | ② | ④ | ⑤ | ⑴a | ⑵a | ⑷a | ⑸a | $s_2$ | $s_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|

| Countermove by "black": | $s_2$ | ④ | ② | (*) | $s_3$ | ⑷a | ⑵a | (*) | ① | ⑴a |
|---|---|---|---|---|---|---|---|---|---|---|

(*) means that an arbitrary countermove is possible

To a further move by "white" to a point in $G_2$, "black" can reply arbitrarily. By assumption (IV.2), "black" can win in this manner.



**Fig. 10a**

**Fig. 10b**

**Fig. 10c**

b) In the case of the graphs in Fig. 10a and Fig. 11a, it is obvious that "black" can win after a move to point ①. Again, this follows from assumption (IV.2).

*About (IV.4), (IV.5), and (IV.6):* to prove these statements, we must consider the possible course of play in the given graphs. To provide a survey of various situations, we will examine the first moves in each case; that is, we will consider the various possible moves "white" has in the graphs in

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Fig. 9a, 10a, and 11a, and black's possible countermoves. Opportunities for counterplay available to "black" in various situations are as follows:

*) "White" can make a connection from $t_1$ into the subgraph $G_2$, or alternatively into the subgraph $G_3$. Only the direct connection to $t$ via the point $d$ is either interrupted or can be obstructed in subsequent play by "black".

**) "White" retains the option to make a connection from $t_1$ to $G_2$. The connections to $G_3$ and the direct connection to $t$ via the point $d$ are either already obstructed by black stones or may be interrupted by "black" in subsequent play. (If in the resulting situation "white" plays on a point from $G_2$ or $G_3$, then "black" can ensure a win on his next move.)

***) The same situation applies as in **), except that "white" can make a connection in the subgraph $G_3$, and "black" can obstruct the other connections.

****) "Black" can no longer prevent a win by "white".

*****) "White" can no longer prevent a win by "black".

As one sees, the assertions in (IV.4) and (IV.5) are correct in cases **) and ***).



Fig. 11 a

Fig. 11 b

Fig. 11 c
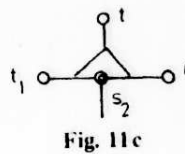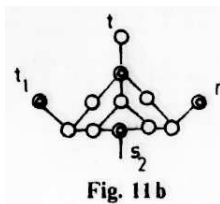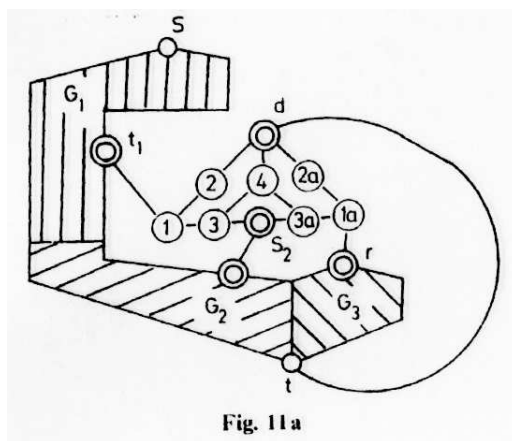
A) "White" has the following choices for moves on the graphs in Fig. 9a:

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*Case 1:* "White" plays on point ①.
"Black" can then counter by:

1.1. playing on point ② *) (this notation means that the situation described under *) applies)

1.2. playing on point ③ **)

1.3. playing on point ④ *)

1.4. playing on any point other than ②, ③, or ④: ****)

*Case 2:* "White" plays on point ①a. This case is symmetric to case 1.

*Case 3:* "White" plays on point ③, ④, or ④a. "Black" then has these options:

3.1. countermove on ①, ②, or $s_2$ **)

3.2. countermove on ①a, ②a, or $s_3$ ***)

3.3. countermove on a still open point ③, ④, ④a, ⑤, or ⑤a *). By a countermove to ⑤ or ⑤a, "black" gives himself additional opportunities to obstruct the play of "white" in the subgraphs $G_2$ or $G_3$, which can also be achieved by playing a countermove directly into points in those subgraphs.

*Case 4:* "White" plays on point ⑤, or ⑤a. *****)

*Case 5:* "White" plays on point ②.
"Black" then has the following options:

5.1. countermove on ① ***)

5.2. countermove on ③ **)

5.3. countermove on ④ *)

5.4. countermove on a point other than ①, ③, or ④: ****)

*Case 6:* "White" plays on point ②a. This case is symmetric to case 5.

B) For the graph in Fig. 10a it is immediately clear that "white" must play his first move on ①, otherwise "black" will play there on the next move, ensuring white's defeat [because of (IV.2)]. Among the countermove options "black" has to a move by "white" on ①, there are essentially four cases:

*Case 1.* "Black" plays on ④. *)

*Case 2.* "Black" plays on ⑤. ***)

*Case 3.* "Black" plays on ⑤ⓐ. **)

*Case 4.* "Black" plays on a point other than ④, ⑤, or ⑤ⓐ. ****)

C) The course of play on the graph in Fig. 11a is, to a certain degree, forced. If a player deviates from the following sequence of play, the opponent has the opportunity to decide the game in his favor on his next move: "white" plays on ①; "black" plays on ②; "white" plays on ③; "black" plays on ④.

*Notation.* We want to introduce notations for the graphs in Fig. 9b, Fig. 10b, and Fig. 11b. The graph shown in Fig. 9b shall be called *white decision graph*, the graph shown in Fig. 10b shall be called *black decision graph*, and the graph shown in Fig. 11b shall be called *meeting-point graph*. To make the following constructions more clear, we will symbolise the graphs shown in figures 9b, 10b, and 11b with the illustrations in figures 9c, 10c, and 11c.

We need a fourth Hex-graph, to substitute for the subgraphs of the "Geography"-graphs constructed in reduction step a) . This graph is shown in figure 12a. (We use the portion of figure 12b that remains after subgraphs $G_1$ and $G_2$ are removed.) We want to consider two different playing situations in these graphs:

(IV.7) *In the graph in figure 12a, "white" is on the move and has exactly one path between the points s and $t_1$ available for playing on in the subgraph $G_1$. Every attempt by "white" to make an unbroken chain of white stones between s and a point in $G_2$ that doesn't pass through point $t_1$ can be hindered by appropriate countermoves by "black". ("Black" is only required to respond to moves by "white" in the partial graph $G_1$.) In this situation, "white" would be forced to play on $s_2$, and "black" has to counter that move with a move to* ②.

In the second play situation the following applies:

(IV.8) *"White" is on the move and has been able to play a complete path between s and r. Every other possible connection between s and t (besides those that connect through r) is either already blocked by black stones or can be blocked by "black" at any time. Then the following holds: If the situation*

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*described in (IV.7) has occurred, then "white" can no longer win. If that situation hasn't occurred, we want to assume that the points* ② *and* ③ *are not yet occupied. In that case, "black" can no longer block the connection betwen s and t if "white" plays on* ① .

*Notation.* We will call the graphs in figure 12b *decision graphs.* We will also call the graphs in figures 9b, 10b, 11b, and 12b *elementary graphs.*
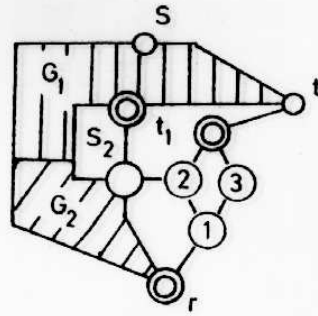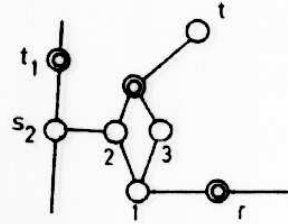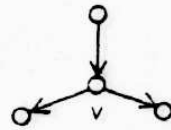


Fig. 12a          Fig. 12b
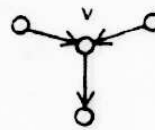


Fig. 13a          Fig. 13b

We can now explain the reduction step b):

In reduction step a), for every given boolean expression $F \in Q$, we produced a bipartite planar "Geography"-graph $G = (V, E)$, with $V = V_1 \cup V_2$ and $s \in V_1$. The first player therefore plays only on nodes from $V_1$ and the second player only on nodes from $V_2$.

By replacing the individual nodes of this "Geography"-graph with white selection graphs, black selection graphs, meeting-point graphs, and decision graphs, we will convert the "Geography"-graph into a Hex-graph.

1. Let $v \in V_2$ and $id(v){=}1$, $od(v){=}2$ (see Fig. 13a). At such a point, the choice of direction for the rest of the game falls to "white". We subsitute such a point with a white decision graph (see Fig. 9b).

2. Let $v \in V_1$ and $id(v){=}1$, $od(v){=}2$ (see Fig. 13a). Here the choice of direction for the rest of the game falls to "black". We subsitute such a point with a black decision graph (see Fig. 10b).

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3. Let $v \in V_2$ and $id(v){=}2$, $od(v){=}1$ (see Fig. 13b). Such a point, which appears in the "Geography"-graph as point $x_{i,2}$ in variable diamonds (see Fig. 2 and Fig. 3) and in intersections as point $a$ (see Fig. 8a), is never a decision point. Therefore we replace such a point with a meeting-point graph (see Fig. 116 [sic]).

4. Let $v \in V_1$ and $id(v) = 2$, $od(v) = 1$ (see Fig. 13b). Such a point, which appears in the "Geography"-graph as point $x_{i,1}$ or $\bar{x}_{i,1}$ in variable diamonds (see Fig. 2 and Fig. 3) or as point $c_i$ ($i = 1,2$) in in intersections (see Fig. 8a). These very points are the decision points. Decision points should be replaced by decision graphs (see Fig. 12b).

5. In the "Geography"-game, the points $v$ in the graph for which $id(v){=}1$ and $od(v){=}1$ have the characteristic of not leaving the opponent different move options after they are played on. The function of these points is solely to alternate the play initiative, that is, the initiative to choose between the two directions from points $v$ with $od(v) = 2$. This assigning of the initiative will be achieved in our Hex-graphs by building elementary graphs. Thus, the function of these points falls away; it will be seen that we won't need to consider them further.

We must now explain how the individual elementary graphs, which should replace the points of the "Geography"-graphs, should b connected with one another.

If two points which are to be replaced by elementary graphs are connected by an unbranched arc move (that is, all points contained in the arc other than the endpoints are incident to only two arcs), then the two elementary graphs are connected, and in particular one of the points $s_2$ or $s_3$ of the elementary graph which stands for the beginning of the arc move is connected with an edge to a point $t_1$ or $r$ of the elementary graph which stands for the end of the arc move. Note that in decision graphs, the graph in the points $t_1$ and $r$ is not symmetric. For decision graphs in the variable diamonds, the point $r$ should be associated with the formula term; for decision graphs in intersections (see Fig. 8a), the point $r$ is associated with the elementary graphs that substitute for $d_i$. [The embedding in a Hex-graph of a "variable diamond" as in Fig. 14a (or intersection as in Fig. 15a) is shown in Fig. 14b (and Fig. 15b, respectively).]

The resulting graph has many points collectively identified as $t$. (The resulting graph is naturally not planar.) In the interconnection of the various elementary graphs, only one point $t_1$ remains which is not connected by an arc with a point $s_2$ or $s_3$ of another elementary graph. This point $t_1$ of the

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elementary graph, which replaces the point $x_{i,0}$ of the "Geography"-graph, becomes the point $s$ of the complete Hex-graph.



Fig. 14a          Fig. 14b



Fig. 15a          Fig. 15b
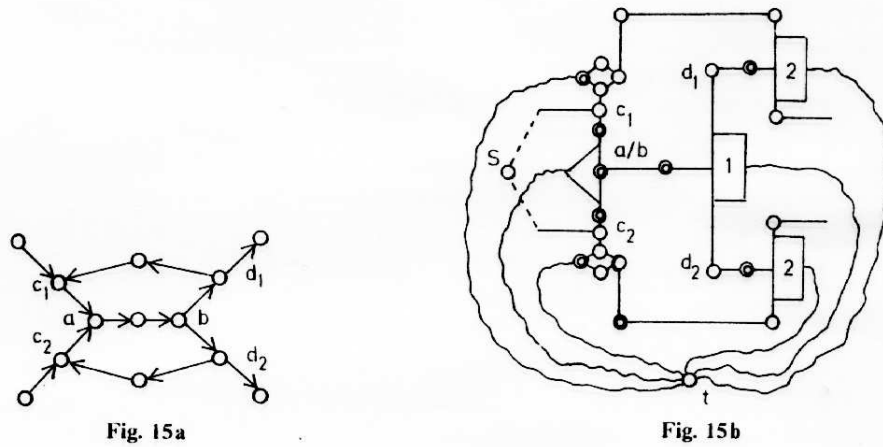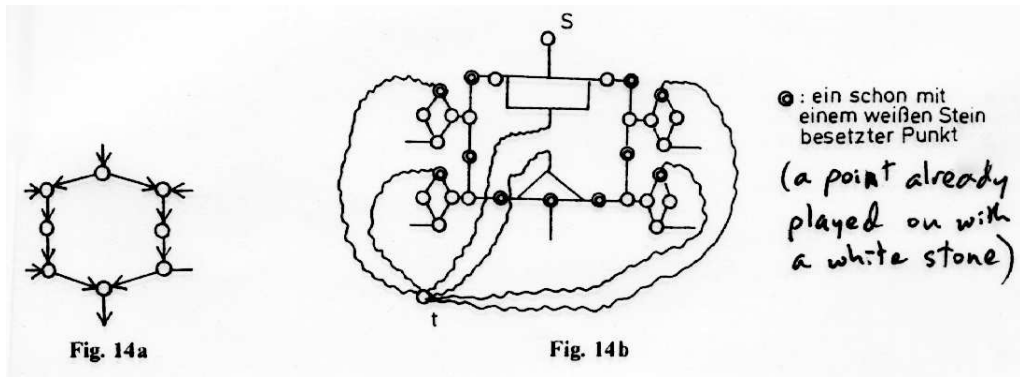
On the now constructed Hex-graphs, Hex is played to a certain degree according to "Geography" rules. The playing situation on these Hex-graphs always matches those of the illustrations in Fig. 9a, Fig. 10a, 11a, or 12a, as they are described in (IV.1) through (IV.8). This is obvious at the start of the game. (The partial graph $G_1$ is then composed only of the point $s$.) Because of statement (IV.4) it can be seen that statements (IV.1) and (IV.2) always remain true when the play crosses over into another elementary graph.

The play on our Hex-graphs proceeds in a manner such that, as already mentioned, "white" lays down a chain of stones starting from point $s$, whose direction "black" can only influence in the black decision graphs. In the following manner, "white" (or "black") has a winning strategy in this Hex-graph, wenn the underlying boolean expression $F \in Q$ is true (or, respectively, false):

(i) If $F$ is true, then "white" can obtain a winning "Geography" strategy in the following manner. In the individual elementary graphs, he always plays to create a connection between the relevant point $t_1$ to one of the points $s_i$ ($i$ = 2,3). In the white decision graphs (see Fig. 9b), he can choose the point $s_i$ himself, and in the black decision graphs the point $s_i$ can be determined by "black" (see Fig. 10b). In the meeting-point graphs and decision graphs there is practically no choice to be made (see Fig. 11b and Fig. 12b). It is clear, how "white" ultimately can win in a decision graph.

(ii) If $F$ is false, then "black" can likewise obtain his winning "Geography" strategy as follows. He just needs to appropriately direct White's play in the black decision graphs. If "white" deviates from the informal "Geography" rules, say by playing in one of the elementary graphs that had already been played in, then according to (IV.2) "black" has the ability to obstruct "white" from making further connections. If "white" plays in an elementary graph in which he has no connection, then "black" can win according to (IV.3).

The "Geography"-game is simulated by a chain of white stones on the Hex-graph.

We would like to explain the reduction step c):

In this reduction step, the not yet planar "Hex-graph" obtained in step b) should be converted into a planar Hex-graph. Of special meaning for our further procedure is the fact that the graph constructed so far is "nearly planar":

(IV.9) *A Hex-graph constructed from a boolean expression according to reduction steps a) and b) immediately becomes planar when all arcs incident to point t are removed. That means, that if the graph is represented in the plane, any arc crossing is between an arc incident to point t and an arc between other points.*

To obtain a planar graph, we change the labeling of the various points of the elementary graph named $t$, referring to them instead as points $t_i$ ($i = 1, .., m$). We introduce a new point $t$, and the points $t_i$ should be connected with this point $t$ by paths. These paths, which we will call *t-paths*, cross the other arcs. We will represent such crossings with yet-to-be-given partial graphs; in so doing it must be guaranteed, that when "white" in some way achieves a connection between $s$ and the point $t_i$ of the most recently played-on elementary graph, "white" can connect this point $t_i$ with $t$ without

difficulty. Otherwise the informal "Geography"-rules retain their validity.

Consider the play situation on the Hex-graph in Fig. 16, for which holds:

(IV.10) *For the Hex-graph shown in in Fig. 16, statements* (IV.1), (IV.2), (IV.3), (IV.4), (IV.5), *and a statement analogous to* (IV.6.3) *hold, so that "white" can force a connection into the graph $G_3$, while on the other hand "black" can interrupt the connection to $G_2$ and the direct connection to t via the point d.*

By the following argument one can be convinced of the validity of (IV.10):

In the situation described, "white" must play on the point ①  if he doesn't want to lose immediately. "Black" on the other hand then has the following options:

*Case 1.* "Black" plays on ② .

After this move the following sequence of play is forced (a player who deviates therefrom can lose at the next move): "white" plays on ③ ; "black" plays on ⑤ ; "white" plays on ④ ; "black" plays on ⑥ .

*Case 2.* "Black" plays on ⑤  or on ⑥ .

In this case, "white" can play on ②  and has a connection in both partial graphs $G_2$ and $G_3$. "Black" must answer this move by playing on one of the remaining free points ⑤  or ⑥ .
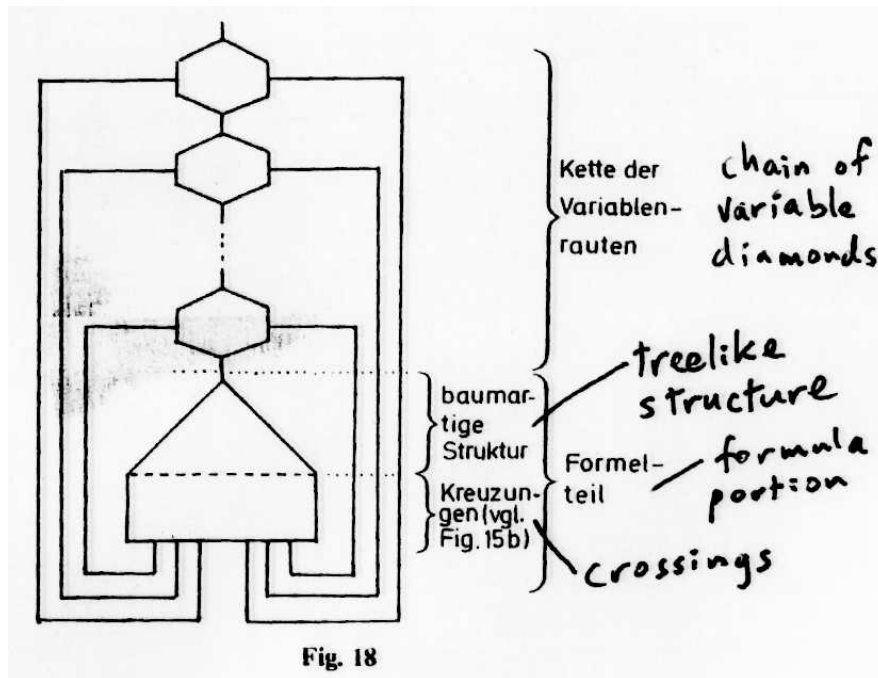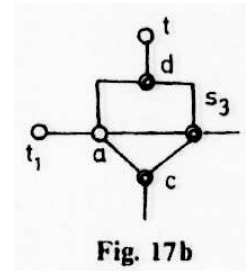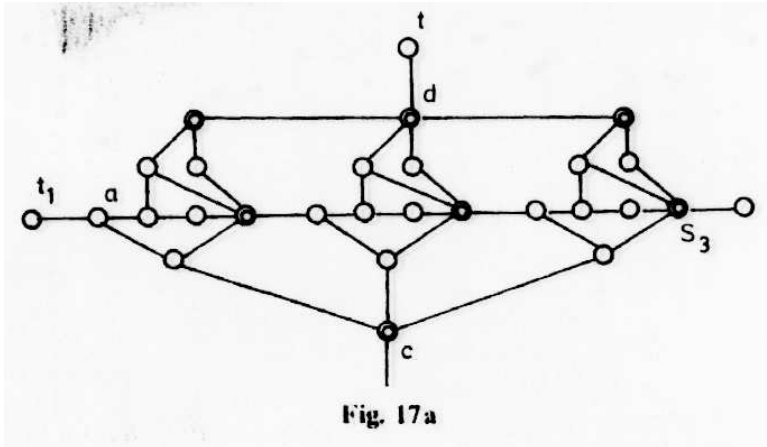
*Case 3.* "Black" plays on a point other than ② , ⑤  or ⑥ .

In this case "black" can no longer avoid losing after "white" plays on ② .

We use the Hex-graphs shown in Fig. 17a to illustrate crossings between *t*-paths and arcs between other points. As one sees, we connect, in sequence, three of the graphs that remain when the partial graphs $G_1$, $G_2$, and $G_3$ in Fig. 16 are removed. We thereby obtain a situation in which "white" can still connect the points c and d as long as "black" can only play two of his stones in these graphs. The graph shown in Fig. 17a shall be called a *crossing graph*. Crossing graphs are the fifth kind of elementary graph we want to use in our construction; in subsequent diagrams they will be symbolized as in Fig. 17b, for the sake of simplicity.

Unfortunately, every possibility for connection between the points c and d in a crossing graph is interrupted when stones are played along the chain

$a$-$s_3$. That means, a $t$-path is interrupted when stones are played along an arc that crosses it. On the other hand, we are free to create several $t$-paths from any point $t_i$ to $t$. It must only remain guaranteed that during the entire play of the game up to that point, when play occurs in one of the elementary graphs belonging to $t_i$, an intact $t$-path from $t_i$ to $t$ exists.



Fig. 17a

Fig. 17b



Kette der Variablen-rauten — chain of variable diamonds

baumar-tige Struktur — treelike structure

Formel-teil — formula portion

Kreuzun-gen (vgl. Fig. 15b) — crossings

Fig. 18

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The $t$-paths shall be constructed as follows. We assume that the Hex-graph constructed in reduction steps a) and b) has the basic appearance sketched out in Fig. 18.

a) $t$-paths $R$ and $L$ are laid down parallel to the chain of variable diamonds. Both paths $R$ and $L$ are directly connected to the point $t$.

b) Each variable diamond is connected to both paths $R$ and $L$ as shown in Fig. 19a.

c) To explain how the points $t_i$ of the elementary graphs of the formula portion are to be connected to $t$ by $t$-paths, we now assume that the "Geography"-graph obtained by reduction step a) didn't have to be made planar due to crossings as shown in Fig. 8a. Crossings as shown in Fig. 15b (as opposed to those in crossing graphs) then play no role in our Hex-graphs obtained by reduction step b).
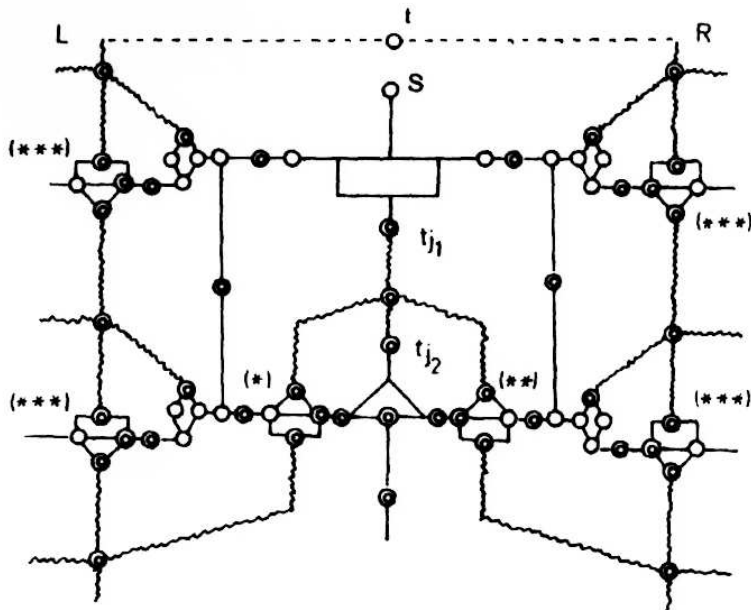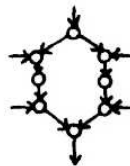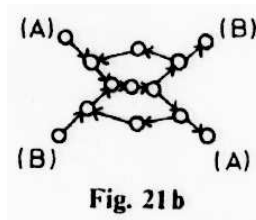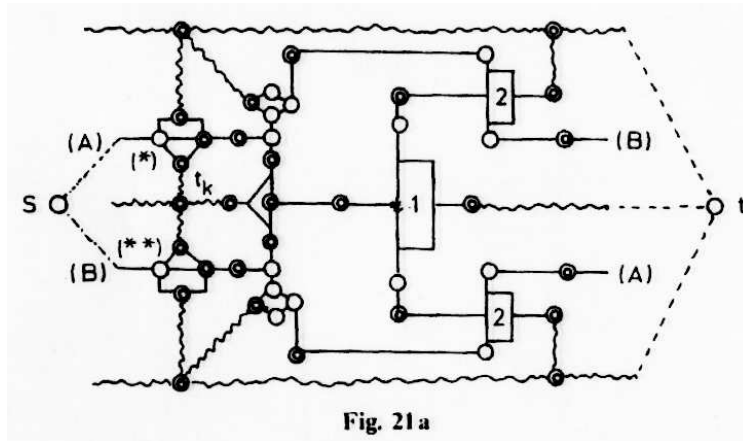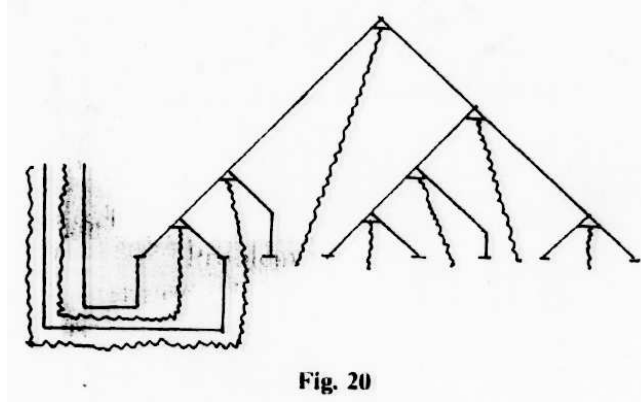


Fig. 19a



Fig. 19b

The formula portion is then in its basic form a binary tree, whose nodes are white or black decision graphs. From the leaves, arcs lead into the chain of variable diamonds. Parallel to each of these arcs we lay down $t$-paths which are attached to the $t$-paths $R$ and $L$. As Fig. 20 makes clear, every node of the tree can be reached via such $t$-paths.



Fig. 20



Fig. 21 a



Fig. 21 b

If reduction step a) required the introduction of crossings as in Fig. 8a, then the problem of crossings between arc-moves which extend out from the

leaves of the "tree" in the formula portion can be resolved in the manner shown in Fig. 21a.



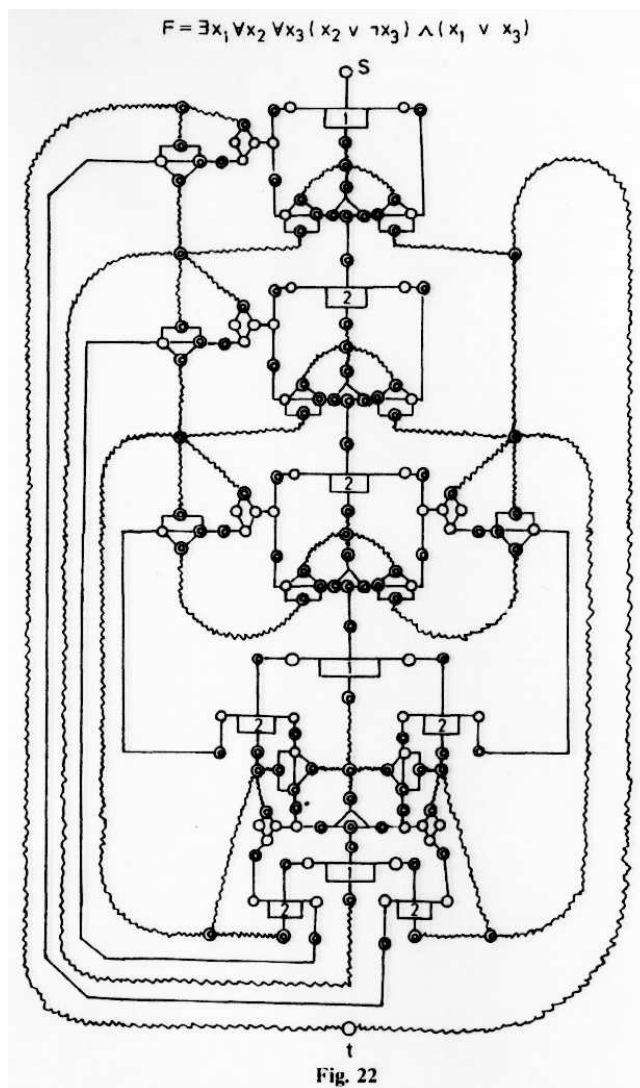$$F = \exists x_1 \, \forall x_2 \, \forall x_3 \, ( x_2 \lor \neg x_3 ) \land ( x_1 \lor x_3 )$$

Fig. 22

The construction described here produces the desired results (compare this to the example in Fig. 22).

(i) One of the common $t$-paths (*) and (**) emanating from points $t_{j_1}$ and $t_{j_2}$ in Fig. 19a must be intact during play in the variable diamonds, because play as a rule only occurs along the right or left half of the diamond. If "white" plays irregularly and tries to play along both halves of the diamond,

that can only work to White's disadvantage. $R$ and $L$ are at this point completely intact in the locations (***), so that an intact $t$-path emanating from $t_{j_1}$ and $t_{j_2}$ exists.

(ii) During play in the formula portion, the crossings of Fig. 21a can no longer be played on. For the same reason, $R$ and $L$ are still intact at locations (***). The points $t_i$ in the formula portion are therefore at the moment in question connected to $t$ by intact $t$-paths.

(iii) In a crossing in Fig. 21a, play can only occur along one of the paths $(A)$ or $(B)$. One of the $t$-paths (*) or (**) therefore remains intact in that location. As $R$ and $L$ are completely intact during play on such a crossing, there exists an intact $t$-path from $t_k$ to $t$.

(iv) If game play ultimately occurs in one of the decision graphs of Fig. 12b, then there must still exist an intact $t$-path via $R$ or $L$ from the point $t_i$ in the graph. $R$ or $L$ is only interrupted at a single location, because play out of a formula portion can only occur along a path. This interruption in $R$ or $L$ lies just outside the portion of $R$ or $L$ that is relevant to the $t$-path from $t_i$ to $t$.

"White" and "black" have practically the same winning strategy on the now planar Hex-graph as they had on the nonplanar graphs, except that play might need to occur along the $t$-paths, on which "white" however can always complete the desired connections.    □

# V

In this section we want to complete the proof of theorem II.1. We need only prove the following lemma:

**Lemma V.1** $G(Graph - Hex) \leq_p G(Hex)$

For the *proof* of the lemma:

The problem of deciding which player has a winning strategy in a given play situation on a Hex-graph can be reduced to the decision problem for Hex, by representing the Hex-graph on a black-and-white grid on an $n \times n$ Hex-board. This is possible, because every not-yet-played-on point on the Hex-graph is incident to at most three arcs.

For such a representation of the Hex-graph, it is necessary to produce an embedding of the graph in the space $\mathbb{R}^2$, which can be achieved by the algorithm given in [4]. Given an embedding of a Hex-graph, it is easy to

obtain an embedding in which the points $s$ and $t$ of the outermost point pair in fact lie outside; thus a Hex-situation can be generated from a Hex-graph in the following manner:

The Hex-configuration has the appearance that on both of White's (goal) sides, all points but one on each side are occupied by black stones. On these points lay white stones, which perform the function of the points $s$ and $t$ (see Fig. 23). The representation of the graph proceeds along the lines of the example in Fig. 24 and Fig. 25. Arcs are represented as shown in Fig. 24, and unplayed-on nodes as in Fig. 24. How occupied nodes of any desired degree are to be represented with white stones is readily evident from the illustrations.

It is clear that in the Hex-situations obtained in this manner, play proceeds as it does in the original Hex-graphs.

If "black" has interrupted all connection opportunities between the white sides, that means that "white" can no longer obstruct a connection between the black sides. □

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