Berge and the Art of Hex

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1 Meeting Claude: my visit to the *MSH*

Claude Berge (mathematician, anthropologist, writer, poet, scuba diver, adventurer) was a man of wide-ranging interests, passions, and activities. One of his passions, certainly one of his favorite pastimes, was the game of Hex.

What I know of Claude, of Hex, and of Claude and Hex, I learned mostly as a result of a six-month visit to Paris in 1984. During that time, Claude and I played Hex on an almost daily basis. Afterwards, I saw him only at math conferences. Even on those occasions, Claude always seemed to have his Hex board with him, and to be eager for a game.

I met Claude in January of 1984. I was a PhD student accompanying my supervisor, Vašek Chvátal, who had just started a half-year visit with Claude; it was during this time that they finished editing their book on perfect graphs [6]. Two of Vašek's other PhD students, Chính Hoàng and Bruce Reed, were also with us (as was Julie Bates, my girlfriend).

Claude was extremely hospitable. In particular, we were always welcome to stop by his office in the *Maison des Sciences de l'Homme (MSH)*, 54 Boulevard Raspail. The office was conveniently located, on Metro lines[1] that we all had easy access to. And so we did stop by, regularly. Claude's office became our *de facto* gathering place, and we soon established a weekday routine: arrive at Claude's office around 11:00, head downstairs for lunch around 12:00, retire to the ground floor bar for coffee around 13:30, return to Claude's office around 14:00, head back to our apartments around 16:30.

During those six months there were many others who frequently stopped by Claude's office, including colleagues Henry Meyniel, Pierre Duchet, Yahya Hamidoune, Michel Las Vergnas, Pierre Rosenstiehl, and Frédéric Maffray, a student of Pierre's.

In addition to being able to talk math and socialize with Claude and the other mathematicians who frequented his office, two things attracted me to Claude's office.

One was lunch.[2] The other was games. Claude had equipped his office with several comfortable chairs and various chess, backgammon, and Hex sets. Whenever a diversion from

* The support of The Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

[1] The nearest stop is Sèvres-Babylon.

[2] Professors in France are civil servants, and *Maison de Science de l'Homme* is a government building. As such, the food in the building's cafeteria is excellent, and plentiful. Each purchaser of a lunch ticket was entitled to a choice of two meats, served with the vegetable and starch of the day, as well as four side dishes, to be chosen from that day's wide selection of cheeses, salads, and desserts. My choice was made simpler by the fact that I always wanted at least one salad, cheese, and dessert; even so, on many occasions I agonized over whether my fourth side dish should be a second cheese or a second dessert.

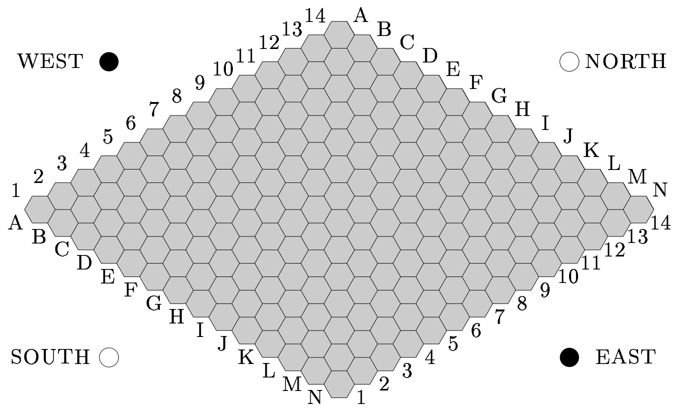

Fig. 1. A hex board. Black's (respectively White's) goal is to form a chain joining the two black (resp. white) sides.

mathematics was needed, a game started.[3] Claude played chess (often with Yahya and Vašek) and backgammon (Vašek, Bruce, Henry, and Pierre were especially keen and usually formed a group game, or *chouette*, which Yahya also often joined), but his favorite game was Hex. He loved to play, and was always happy to introduce others to the game. Thus it was that Bruce and Chính and Julie and I learned about Hex from Claude.

2 Claude and Hex

Hex was invented by Piet Hein in Denmark in 1942 and independently by John Nash in Princeton around 1948.[4] While Claude was based in Paris, he also spent the 1957-58 academic year in Princeton, and so could have learned of the game from either source. It was most likely during his time in Princeton in the 1950's that he was introduced to Hex. Claude was interested in game theory and Nash had just established the game theory result which was many years later (in 1994) to win him a Nobel prize, namely, that for any (reasonably defined) multi-player game, there exists at least one equilibrium point in the strategy space [14]. At that time, Hex was often being played by Nash, David Gale and others in the common room of the Princeton math department, and anyone visiting the department would have been exposed to the game.

The game takes its name from the playing board, which consists of a diamond shaped array of hexagons as shown in Figure 1. I do not know how often Claude played Hex initially,

[3] Actually, with all the game playing that went on in Claude's office during this time, it was rather the case that whenever anyone wanted to work uninterrupted, they went elsewhere. Claude worked at home a lot during those six months.

[4] Some credit should also go to David Gale, at the time a Princeton instructor. Nash's initial formulation of the game was purely abstract (he did not build any playing apparatus) and somewhat awkward. It was Gale who, upon learning of Nash's game in early 1949, came up with the ideal board shape (the same diamond-shaped array of hexagons used by Hein), Gale who built such a board, and Gale who then donated the board to the Fine Hall (math department) common room. Its frequent use soon spread the popularity of the game [15]. Harold Kuhn, a fellow Princeton graduate student of Nash's, cites a more awkward original formulation of Nash's game as evidence that Nash's conception was independent of Hein's [12].

but it seems that it was not until around 1970 that he acquired his first Hex board.[5] Michel Las Vergnas, a longtime colleague and former student of Claude's, recalls coming across some teak Hex sets in a Danish specialty shop[6] near *Place de l'Etoile* in Paris. Each set consisted of a board with a twelve-by-twelve array of holes, together with two bags of wooden pegs (to make a move, a player inserted a peg into a hole). Michel bought a set and showed it to Claude, who promptly bought one for himself.

After these purchases, Claude, Michel, and Jean-Marie Pla (another colleague) played Hex regularly at the *MSH* during their coffee breaks.[7] They soon reached a point where the outcome of a game was usually decided after only a few moves. Thus it was that Claude was motivated to construct a larger fourteen-by-fourteen board.

Claude's first fourteen-by-fourteen creation was obtained by drilling fifty-two extra holes in his twelve-by-twelve board. The end result was unsatisfying, much to Michel's chagrin, as the beautiful wooden board had been effectively destroyed. Claude, however, was not bothered. He found the process of inserting pegs into holes cumbersome, and also found the heavy boards inconvenient for travel. Claude thought plastic would make a more suitable construction material.

In fact, Claude had in mind an ambitious project: the commercial production of Hex sets. As a first step, a half dozen prototype boards were fabricated (including one black, one silver, one pink), with shallow pits at the hexagon locations and small beads for playing pieces. The black board became Claude's. He obviously found it suitable for travel, as he took it with him on virtually all his future trips.[8] The silver board went to Michel, who still uses it. The pink board was found to be distracting to use, and has since vanished.

Unfortunately, Claude's project was never completed; the commercial production of Hex sets never came to pass. Claude once remarked that he had been unable to find a supplier who could provide beads at reasonable cost; I do not know if this was the only factor. However, along with the prototype boards, another important aspect of the project survives: a draft of *L'Art Subtil du Hex* [3], an introduction to the game that Claude had planned to include with each game set.[9]

Consisting of about twenty half-sized pages, presumably so that it could be printed as a small pamphlet, *L'Art Subtil du Hex* is surprisingly comprehensive, with sections on Hex and mathematics, strategy, and puzzles as well as the game's rules and origins. The article

[5] As the shape of a Hex board is not shared with other commonly played games, acquiring a Hex board requires some doing. Many players end up making one.

[6] The boards were in a Danish boutique because they were made in Denmark by Skjøde of Skjern, a Danish company formed by Piet Hein, the original inventor of Hex. This was explained in an accompanying brochure, which also claimed that Einstein had played the game. The claim is probably true, as in the late 1950's Einstein was at Princeton's Institute for Advanced Studies and would have had frequent contacts with the Math Department.

[7] The photo accompanying this article, taken by Michel, shows Claude and Jean-Marie playing while Neil Grabois, another research visitor to the *MSH*, looks on.

[8] The board will likely be used for many years to come, as before he died Claude passed it on to César Bondy, the son of Claude's friend and colleague Adrian Bondy.

[9] On my photocopy of the typed manuscript, the following annotation appears in Claude's handwriting (on the top of the first page, above the title "L'ART SUBTIL DU HEX"): *Introduction au jeu du Hex – extrait de la préface à la première version commerciale du Hex 14 × 14 (1977).*

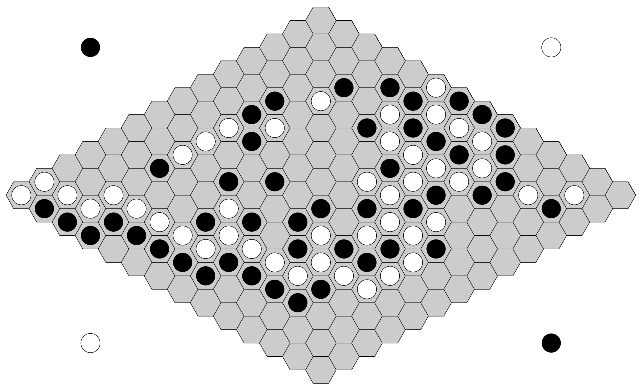

Fig. 2. White has a chain joining the two White sides, so White wins.

expands and updates Martin Gardner's introduction to Hex [11], the principal additions being Claude's commentary on various aspects of the game (ideal board size, handicapping, advanced strategy), as well as several new puzzles and their solutions. Just as Gardner's introduction to Hex had popularized the game in North America, so I believe that Claude was hoping that his pamphlet would popularize the game in France.

As Claude's pamphlet was never published, for the benefit of those readers who wish to read Claude's commentary for themselves I have included a translated (and slightly revised[10]) version of *L'Art Subtil du Hex* in the next section.

3 *L'Art Subtil du Hex*

Rules. *There is no book*[11] *describing in a precise manner the rules and strategic principles for playing Hex. This game, practically unknown in France, is nonetheless considered by many as the game which best lends itself to subtle analysis, to surprising combinations, to unexpected reversals of fortune.*

The rules are simple: two players in turn repeatedly place a stone on an unoccupied hexagon of the diamond shaped board[12] *The player who starts – say "White" – places a white stone, the other player – say "Black" – then places a black stone, and so on. Once a stone has been placed, it is never moved. White's goal is to form a white chain (or path) joining the two parallel North and South sides of the board. Black's goal is to form a black chain joining the two other sides, West and East. For example, in Figure 2 there is white chain joining the two white sides (which Black can never "cross" to connect the two black sides), so White wins.*

[10] This translation is based on an annotated copy of the manuscript [3] on which Claude had marked one minor revision and one correction. The annotated draft is surprisingly polished, given that Claude never signed off on it; other than correcting a few typos it was necessary for me to make only one further revision (in the solution to Puzzle 5). Figures 2 through 7 are taken directly from [3].)

[11] The first and to date only book on Hex did not appear until 2000 [7].

[12] for example, as in Figure 1.

In Figure 3, on the other hand, White is doomed. While Black has no winning chain, White cannot prevent Black from forming one. Black has a "virtual connection" from the West to B13, consisting of the two intermediate locations marked u; if ever White plays in one location, Black can play in the other. Similarly, the location sets marked v, w, x, y, z *form further Black virtual connections which, together with interconnecting Black chains, form a side to side (and so winning) virtual chain for Black.*

Each of the four corner board locations belongs to both of the sides it intersects. Thus the board in Figures 1 through 3 has exactly fourteen hexagons along each side. This is the ideal size for a Hex board, since experience shows that on smaller boards a strong player who first occupies a central location almost always wins. If two players are unequal in ability, the stronger can give a handicap advantage to the weaker, by allowing the latter to place one, two, or even three stones to start the game, and each player will end up with a challenging game. Beginning players can learn the game by playing on a smaller board, say five-by-five (or "Mini-Hex"). Such a board can be formed by simply marking off the boundary on a full sized board, as shown in Figure 4.

Origins of the game. *Unlike chess and go, whose origins are cloaked in the mists of antiquity, Hex was discovered in the middle of the twentieth century. Hex was invented*[13] *in 1942 by the Danish physicist*[14] *Piet Hein who introduced the game at the Niels Bohr Institute in Copenhagen under the name of "Polygon"*[15] *and independently in 1949 by the American mathematician John Nash, at the time a student at Princeton, who popularized the game in the USA, where it was initially called "Nash". Nash disappeared prematurely from the mathematical scene,*[16] *but not before having discovered an important theorem in game theory and an equally essential theorem in differential equations.*

Hex is also closely related to a board game invented by the father of information theory, Claude Shannon. The study of the so-called "Shannon switching game" has led recently to interesting developments in matroid theory.[17] *Unfortunately, a winning strategy for this game has been discovered,*[18] *and so it is no longer interesting to play.*

Hex and mathematics. *Pieces never move once played; the rules are simple, and the game can be played at any age. One might therefore wonder why Hex seems so interesting to mathematicians. First and foremost, and this was discovered by Nash, there exists a winning strategy for whoever plays first. However, as the proof is non-constructive, it reveals nothing about such a strategy (other than that it exists); no one knows of any such winning strategy.*

[13] It is interesting that Claude uses both *discovered* and *invented* in referring to the origin of Hex. I wonder which term he felt was more accurate, namely whether Hex was a natural mathematical artefact waiting to be discovered, or rather a result of human creation.

[14] and engineer, writer, and poet. Like Claude, Piet Hein was a man of many interests and talents.

[15] According to John Milnor, Hein also referred to the game as Hex [14].

[16] Nash suffered from schizophrenia. The story of his life, including his unexpected recovery from this illness and the winning of a Nobel prize, is documented in Sylvia Nasar's excellent biography [15].

[17] See [13].

[18] See [13].

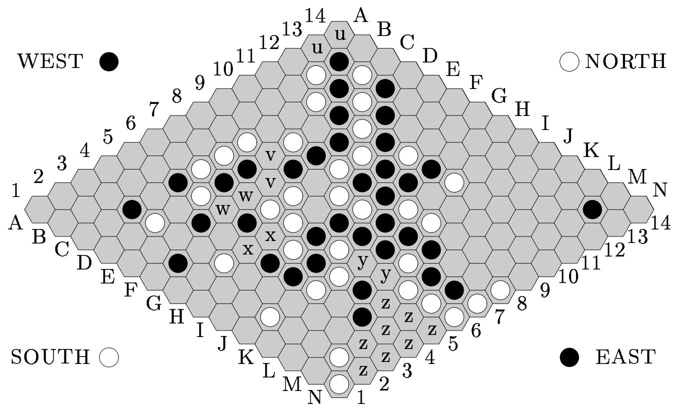

Fig. 3. Regardless of whose turn it is to play, Black has a virtual chain joining the two Black sides, so White is doomed. From the West, the virtual chain passes through the set of locations marked u, namely {A13,A14}, to the black stone at B13, and then follows a black chain which passes in turn through the sets of locations marked respectively v, w, x, y, and z, reaching the East.

Nash's proof is roughly as follows. First, the game cannot end in a draw,[19] *so it follows by the Zermelo-von Neumann theorem that either the first or second player has a winning strategy. Assume that there exists a winning strategy for the second player (say Black). Under this assumption, the first player (White) can play as follows: for the first move, play anywhere; for subsequent moves, ignore the initial move (and so pretend that Black moved first) and follow Black's (winning second-player) strategy. White's required moves will always be either to an unoccupied location or to a location which already has a white stone, in which case White can play in any unoccupied location. It follows that White can follow Black's strategy, and so White (as well as Black) has a winning strategy. But only one player wins a Hex game; this contradiction completes the proof.*

Those readers who desire a more formal proof than that given here can consult [5], where we extend this kind of reasoning to a larger class of positional games.

On a five-by-five board, it is easy to show that the first player can win in at most seven moves by playing first in the centermost location. For larger boards, it also seems advantageous to start in a centermost location. To offset this advantage, players often introduce an extra rule restricting the first player to play near one of the two acute corner locations (A1 or N14 on the fourteen-by-fourteen board).[20] *In fact, a mathematician from Wisconsin, Anatole Beck, has shown (using a non-constructive proof similar in flavour to Nash's) that the second player can win if the first player starts in an acute corner [1]. More recently, with*

[19] At most one player can win. This is easy to see, as every set of locations which forms a white winning chain intersects every set of locations which forms a black winning chain. Much more surprising is that *at least* one player will win. Regardless of how any collection of black and white stones is placed on a Hex board, a black or white winning chain will eventually appear. Gale showed that this topological property (which can be proved by induction on the size of the Hex board [1]) is equivalent to the celebrated Brouwer fixed-point theorem [10].

[20] Once I was good enough to play Claude on even terms, we followed the convention that the first player's move must be within two locations of an acute corner. Claude's most common opening was B2.

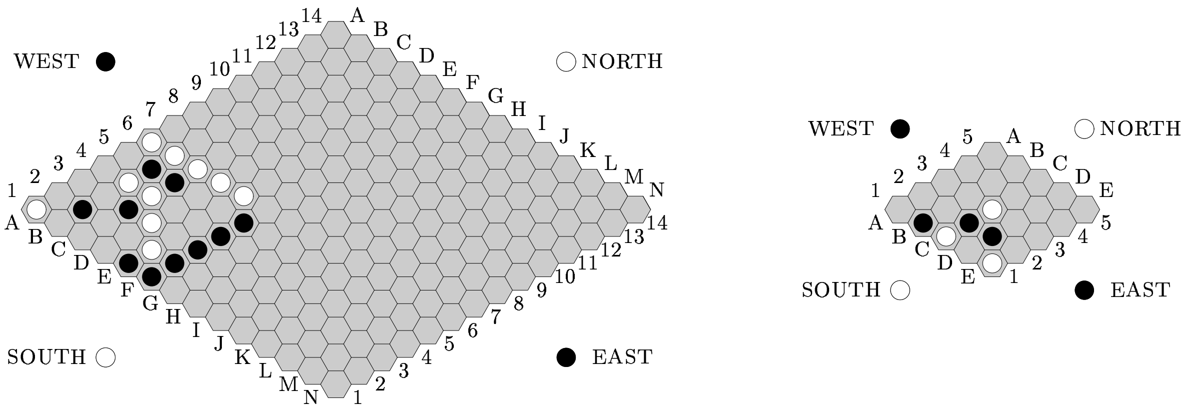

Fig. 4. Puzzles 1 and 2 (Mini-Hex). In each, White to play and win. In Puzzle 1, notice how a smaller board is formed from a larger one. Puzzle 2 is originally due to Piet Hein and also appeared in [11].

similar arguments Beck and Charles Holland have shown that with [1. A1 A2] the first player regains the advantage and can win, and that with [1. B1] the first player loses [2].[21]

It is also interesting to play the misère or "whoever wins loses" version of Hex (also known as Reverse Hex) in which the player who first forms a connecting chain loses the game. For this game it is known that there is a winning strategy for the first player if the board size is even, and for the second if the board size is odd [9][22]

Some strategic principles. *In Hex, as in other games such as chess, there are different styles of play. Aggressive play consists of placing a stone where it threatens to form one or more connections. Defensive play consists of placing a stone where it can most easily cut the most threatening of the opponent's connections as soon as they appear. These two strategies eventually come together, since the only way to block all opponent threats is to form one's own connections.*

Beginning players will to their detriment often play in a shortsighted manner, attempting only to lengthen their most promising chain: they feel as though they are "attacking". Experienced players, on the other hand, will play their stones in a more dispersed fashion, and by combinations of "double threat" moves eventually form a connection.

A theoretical principle which is essential for all players to be aware of is the following: a virtual chain which extends to the third row from a side (such as the black virtual chain which reaches L4 in Figure 3) can be extended to the side, as long as the locations comprising the extending triangle (M3, M4, N2, N3, N4) together with one of the two adjoining columns (L3, M2, N1 or L5, M5, N5) are unoccupied.

Similarly, the chain can be extended to the side if the only occupied location in the extending triangle and the two adjoining columns is directly opposite the stone (N3).

There are analogous rules for chains which reach the fourth row from a side, but rather than describe them, we will illustrate some of these strategic principles with some puzzles.

[21] The reference cited in [3] for this result, namely "A. Beck, Ch. Holland, *More on Hex* (to appear)", never actually appeared. However, the result is included in [2].

[22] Gardner gives an earlier attribution to Robert Winder [11].

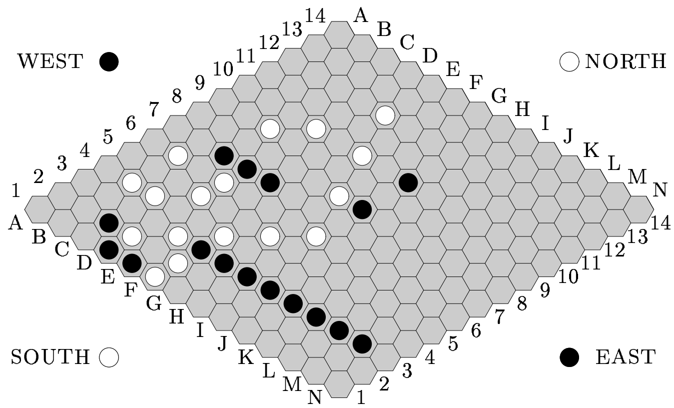

Fig. 5. Berge's Puzzle 3. Black to play and win.

Puzzles. *The greatest chess puzzle composer of all time, the American Sam Loyd, said that his goal was to compose puzzles whose solutions require a first move that is contrary to what 999 players out of 1000 would propose.*[23] *This criterion also applies to Hex: in many puzzles, the solution seems paradoxical. We begin with an easy five-by-five puzzle.*

Puzzle 1 (Figure 4, left). White to play and win. Here the winning move is unique. Begin by observing that the white chain can easily connect to the North, due to the double threat A5 and E4. But how can the chain join the South?

We use chess notation to describe move sequences, with each move number followed by the board coordinates for White's move and then Black's. Question marks and exclamation marks indicate weak and strong moves respectively. For example, if White tries [1. A2?] then Black can reply with [1. – B3!] and White is doomed, since Black is virtually connected to the West, and also to E1 by an easy sequence of double threats.

If White does not immediately see the winning move (which we remind the reader is unique), he or she can always adopt a defensive mentality. If White does not play in any of A2, A3, C1, C2, D1, Black can form a winning virtual chain on the next move with D1. Thus White has at most five moves to consider. (The solutions are given in the Appendix.)

Puzzle 2 (Figure 4, right). White to play and win. Again there is a unique winning move, this time more difficult to find.

Puzzle 3 (Figure 5). Black to play and win. Here White has an impenetrable wall from F1 to D14. But Black can break through. How?

Puzzle 4 (Figure 6). Black to play and win. The same idea appears here. The two black chains seem completely cut off, one by a white wall from A11 to N14, the other by a white wall from B8 to N1, and the two sets of gaps from each wall do not intersect. It thus seems as though Black's two chains are completely useless. And yet ...

[23] Claude also mentioned this quotation in his tribute article to Martin Gardner [4].

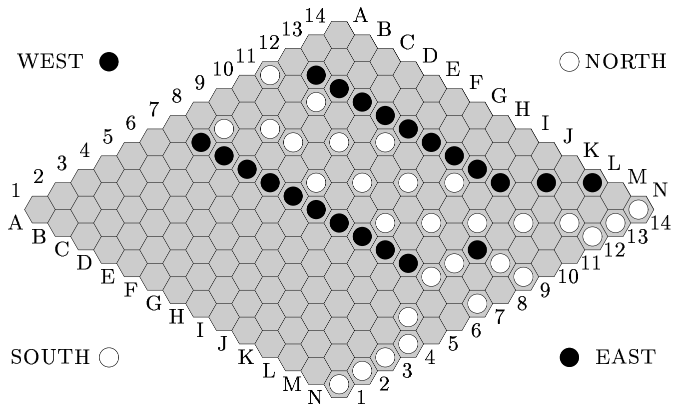

Fig. 6. Berge's Puzzle 4. Black to play and win.

Puzzle 5 (Figure 7). White to play and win. This is a study rather than a puzzle. The interesting question is whether White can connect G11 to the North.[24] *Black ends up trying to connect G12 to the East, and there are many variations to consider.*

Finally, in playing the game for his or herself, the reader will undoubtedly discover new Hex problems whose solutions require precise and unexpected play.

Claude Berge

Appendix: Puzzle Solutions.

Puzzle 1. The winning move is [1. C2!]. One might also consider [1. A2 (or A3) B3!] or [1. C1? (or D1?) D2!], but these both lead to winning Black virtual connections.

Puzzle 2. The winning move is [1. B3!], for now either C1 or E1 (via E2) extends victoriously to the North. All other White moves lose. For example, [1. E2? D3!] (and not [1. – D4?], which leads to [2. B3!], and the two central white stones are connected to the South by a double threat, and to the North by a "third row" virtual connection). Also, [1. D3? E2!] and [1. B2? B3!].

Puzzle 3. Black can play [1. – F5], threatening [2. – G4]. The intermediate moves E6, F6, B7, . . . serve only to delay matters, and eventually White must respond with G4, in which case all White responses are forced, with [2. G4 E4!], [3. E5 D4!], [4. C5 D3!], and the black connection is now evident.

Notice that this position would be unlikely to occur in an actual game (allowing Black to form such a chain would be a colossal blunder for White). This does not concern us, as we are interested in constructing interesting puzzles rather than replicating realistic game

[24] The other obvious question is whether White can run from C12 to the South. Black can complicate matters considerably by well timed moves to E4 and E3.

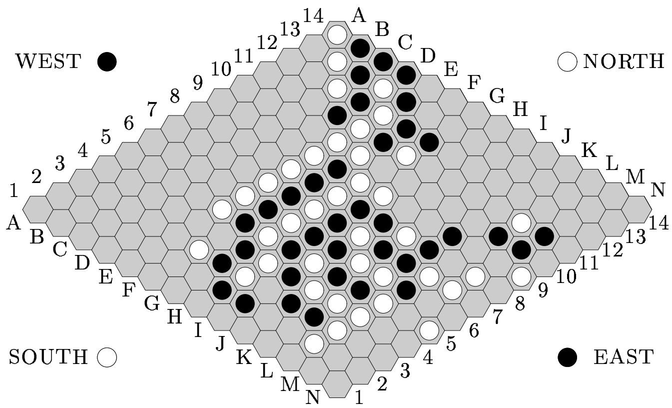

Fig. 7. Berge's Puzzle 5. White to play and win.

situations. For this reason we do not insist that the number of black and white stones should be equal (one player may have started with a handicap). Also, we do not insist that our puzzles admit unique solutions (another winning move for this puzzle is [1. – E4!]), as such a restriction would eliminate any possibility of interchangeable intermediate moves, which often adds to the beauty of a solution.

Puzzle 4. The isolated black stone at L9 can actually be connected to the West through the constellation of white stones. First, though, two intermediate moves are useful: [1. – M6], [2. L6 M11], [3. L12]. These two Black moves are interchangeable, and White has no better response. The Black threat is very precise, and any intermediate White responses change nothing. White's responses are now forced: [3. – L10!], [4. K11 K9], [5. K8 J10], [6. J11 I10], [7. H11 I9], [8. J8], and so on. Black's chain snakes slowly through the White stones from L9, ending eventually on the West side at A9. The connection to the East is assured because of the initial two moves.

Puzzle 5.[25] *The key is to maintain as long as possible the double threat of joining the stone at L11 with either the stone at G11 or the stone at M8 (the latter can easily be joined to the South).*

For example, one winning move is [1. H11], with a typical continuation being [1. – H12] [2. I11 I12] [3. K11!].[26] *The white stones at K11,L11 are now virtually joined to the South (White can respond to [4. – J11] with [5. K10!], threatening both J10 and L9), and also to*

[25] This is a revised version of the solution in [3], which overlooks [1. K10 I11].

[26] Here are the main lines of play.

H11	G14	I12	I13	J12								
"	H13	I12	I13	J12	J14	J13	I14	L13				
"	H14	H12	G14	I13	I14	K13	J13	K11				
"	I12	H12	G14	H13	H14	I13	I14	K13	J13	K11		
"	I13	I11	I12	K11								
"	I14	H12	G14	I13	H14	K13	J13	K11				
"	H12	I11	I12	K11								
"	"	"	I13	K11								
"	"	"	I14	I12	H14	K13	J12	K11				
"	"	"	J12	I12	H14	I13	I14	K13	J13	K11		
"	"	"	J13	K11	J11	I13						
"	"	"	J14	I12	H14	J13	I14	L13	K13	L12	J12	K11

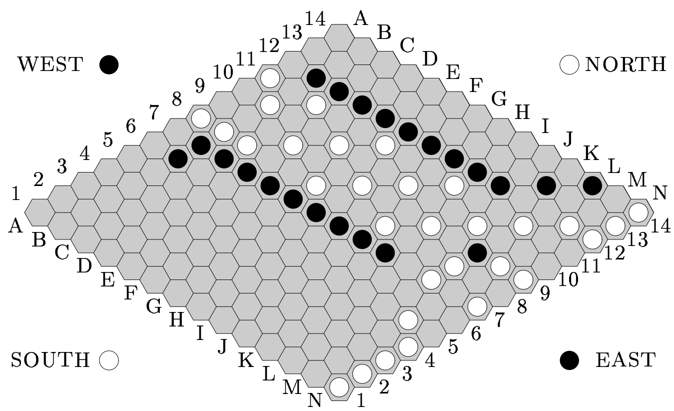

Fig. 8. The corrupted Berge puzzle, as it appeared in *The Mathematical Gardner* [4]. There was supposed to be a black stone at K7. As Browne pointed out in a detailed analysis in [7], Black now has no winning play.

the North (White can reply at J13 to any Black move to a location not in {J12, K12, J13, I14, J14}, and at L12 to any Black move to a location not in {L12, M12, K13, L13, M13, J14, K14, L14, M14}; the intersection of these two must-play regions is J14, and if Black moves there then [5. – J14] [6. K13 K14] [7. M13!] wins).

The move [1. K10?] is also tempting, since if Black decides to eliminate the White double threat with [1. K10 L9?] then White can reply with [2. H11!], which leads to a White win[27] *But Black has a better reply, namely [1. – I11!], and now White loses.*

4 The case of the missing stone

One small part of *L'Art Subtil du Hex* was eventually published, albeit in a form which unexpectedly caused Claude considerable frustration. In 1981, Claude contributed the article "*Some Remarks about a Hex Problem*" to "*The Mathematical Gardner*", a book edited by David Klarner in tribute to Martin Gardner [4]. The problem in question, shown in Figure 9, is a minor variation of Puzzle 4 from *L'Art Subtil du Hex*; the article contains a brief discussion of Hex, the presentation of the puzzle, and an analysis of its solution.

Unfortunately, the diagram appeared in [4] with a black stone missing, as shown in Figure 8. As misfortune would have it, the absence of the stone changes the outcome of the puzzle, and there is no longer a winning move for Black.

Few readers were aware of the omission, and many noticed that Claude's analysis is not correct for the puzzle as pictured. This was the case with Cameron Browne, who in [7] gives a detailed analysis showing that Black has no winning move for the puzzle as shown in Figure 8.

Of course, Claude's analysis in [4] *is* correct for Figure 9, the figure that was *supposed* to appear in the article. Browne himself pointed this out as soon as the matter was brought to his attention by Renaud Palisse, a student of Claude's [8]. The reader can also verify this by following Claude's analysis of the solution to Puzzle 4 in the previous section.

[27] It is a nontrivial exercise to show this. There are many lines of play to consider.

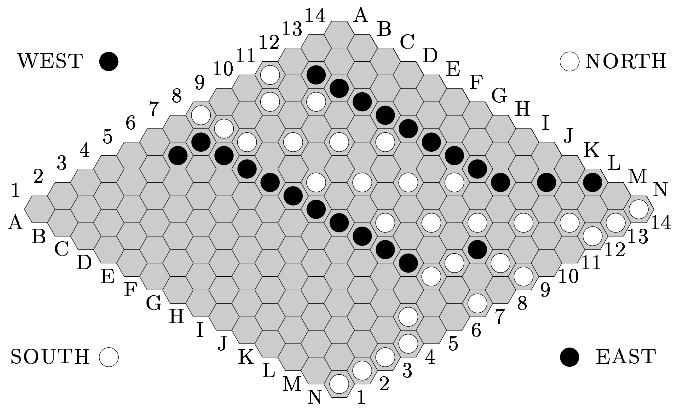

Fig. 9. The figure that was supposed to appear in *The Mathematical Gardner* [4]. Black to play and win.

5 More strategy

L'Art Subtil du Hex was intended to be no more than a Hex primer, and so has little discussion of advanced strategy. Even so, it at least touches on (either explicitly, or implicitly via the puzzles and their solutions) three key strategic concepts that I learned from Claude: virtual connections,[28] double threats,[29] and lookahead (looking ahead to consider the most likely continuations).

While lookahead is of course useful in any two-player game, a particular form of it is well suited to Hex endgame analysis, as Claude explained to me after I had grasped the rudiments of the game.

Near the end of a closely fought Hex game, each player will be close to having a winning chain, and so a blunder by either player will lead to an immediate virtual chain for the opponent. In such situations, it is advisable to consider all possible opponent moves before making one's own move. In particular, every opponent move which results in the creation of a winning (for the opponent) virtual chain must be refuted. Thus, to be assured of not losing the game, it is necessary for a player to play in a location which is in the intersection of all of the opponent's potential winning chains. The key point is that this intersection might be relatively small in size, and so only a small number of moves need be evaluated.

To illustrate, consider Claude's Puzzle 4 (in Figure 6), where Black will lose unless playing carefully. In particular, if Black passes (skips a move, and lets White play), White has several winning moves. One is M6, which yields a winning virtual connection {N6, M7, M8, N8, L10, M10, L12, M11}. Consequently, we call {M6, N6, M7, M8, N8, L10, M10, L12,

[28] Formally, a virtual connection is any set of locations which "virtually connects" a pair of target locations for a player, in the sense that, even if the opponent has the next move, the player can always connect the two target locations. The two kinds of virtual connections shown in Figure 3 (those consisting of exactly two unoccupied locations, and those which connect to a side) are especially common, so learning to recognize them is crucial.

[29] Claude was always looking for double threat moves. One of his favourites was to play a move such as B2 at the appropriate time. For example, if Black has a chain running near the North from East to West, and White moves to block the black chain by playing in the Northwest, then B2 threatens both to join a southerly extension of the Black chain and also to start a new chain heading to the East near the South.

M11} a *potential win-set* for White, since if Black's next move leaves all locations in this set unoccupied, then White wins.

Another potential win-set for White (created by playing at N6) is {N6, N5, M6, M8, N8, L10, M10, L12, M11}. A third (via K9) is {K9, L6, M6, L10, K11, M11, L12}.

It follows that Black's next move must intersect *each* potential win-set for White, for otherwise White can convert one of the potential win-sets into a winning virtual connection. Thus Black's "mustplay" region for the next move can be restricted to the intersection of all potential win-sets for White, namely one of the moves in {L10, L12, M6, M11}. Thus Black need only consider these four moves.[30]

6 Final Pieces

I had a lot of fun playing Hex with Claude. The games were invariably challenging and the ambience convivial. Claude played with an infectious, joyful manner; he was forever saying "aaaaaaaHA!" with a twinkle in his eye when placing a stone, as though that move was about to break the game wide open. And spending all that time with Claude gave me plenty of opportunity to quiz him about the many other interesting facets of his life.[31]

Claude was of course interested in Hex because of its mathematical connections. He was always wondering whether someone would come up with some mathematical insight which would help win the game. Most of all, though, I think that Claude was interested in Hex because he just loved to play the game.[32]

Thanks for all the games, Claude. I leave the final words to you.

[30] As Claude's analysis for Puzzle 4 shows, Black can win by playing either M6 or L11 first, and then playing L10. Black can also win by playing L10 first. However, White can win if Black plays L12.

[31] I was particularly intrigued by his adventure stories. Claude was an avid swimmer, and two tales from his many tropical trips come to mind.

On one occasion, while snorkelling in a lagoon after having been dropped off on a small uninhabited island, Claude noticed a bright green light glowing in the water below, and swam down to investigate. By the time that he discovered that the light was coming from an opening in the coral reef, he was being swept through the opening. Claude attempted to turn back, but the current was too strong, and he was swept down through the reef wall and out to sea. Upon surfacing, he realized that he was in riptide, and so he swam perpendicular to the current, coming eventually to a spot where it current weakened and he was able to swim back to the island. He discovered later on that several people had drowned while swimming near that island.

On another occasion Claude was swimming alone offshore when he found himself in the company of two sharks, one on each side. The sharks swam parallel to Claude, moving gradually ever closer in a way that seemed threatening. When one shark neared to within an armlength, Claude moved his hand in a sudden motion toward the shark, as though he were about to strike it. The shark responded by giving Claude a bit more distance. Claude repeated this action several times during his swim back to shore, which ended uneventfully.

Birgit Bock recalls the preceding episode (which occurred during a trip to Kenya) as well as another occasion. Claude was diving and called to Birgit to join him, as he had spotted a strange looking fish with feathery floating appendages. They got very close to the fish. Later on, Claude learned that it was a lionfish, an extremely venomous fish. As Birgit points out, Claude liked danger and lived intensely.

[32] Bruce Reed recalls the following scene taking place on several occasions late at night after he and Claude had played several games of Hex.

Claude: "One more game? One more game?"

Birgit Bock (Claude's girlfriend), stamping her foot: "Mais non, Claude!"

Claude: "Ah, just *one* more game ... "

It would be nice to solve some Hex problem by using nontrivial theorems about combinatorial properties of sets (the sets considered here are groups of critical holes). It is not possible to forget that a famous chess problem of Sam Loyd (the "comet"), involving parity, is easy to solve for a mathematician aware of König's theorem on bipartite graphs; also in chess, that the theory of conjugate squares developed by Marcel Duchamp and Alberstadt is a beautiful application of the algebraic theory of graph isomorphism (the two graphs are defined by the moves of the kings).

The use of a mathematical tool may be unexpected and therefore adds some new interest to a game; but Hex exists as a most enjoyable game in its own right for mathematician and layman alike.

Claude Berge [4]

Acknowledgements. I am indebted to Birgit Bock, Adrian Bondy, Vašek Chvátal, Renaud Palisse, Bruce Reed, and Michel Las Vergnas for contributing information on Claude. I also thank Adrian, Michel, Vašek, Nathan Po, Glenn Rhoads, and Jack van Rijswijck for detailed commentary on earlier versions of this article, and Nathan, Yngvi Björnsson, Mike Johanson and Morgan Kan for assistance in implementing a computer program which was used to verify some of the lines of play.

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